

The critical group of $K_m \times C_n$ *

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ABSTRACT

In this paper, the structure of the critical group of the graph $K_m \times C_n$ is determined, where $m, n \geq 3$.

Keywords Graph; Laplacian matrix; Critical group; Invariant factor; Smith normal form; Tree number.

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1 Introduction and statement of results

The critical group of a connected graph is a finite abelian group whose structure is a subtle isomorphism invariant of the graph. It is closely connected with the graph Laplacian.

Let $G = (V, E)$ be a finite connected graph without self-loops, but with multiple edges allowed. Then the Laplacian matrix of G is the $|V| \times |V|$ matrix defined by

$$L(G)_{uv} = \begin{cases} d(u), & \text{if } u = v, \\ -a_{uv}, & \text{if } u \neq v, \end{cases} \quad (1.1)$$

where a_{uv} is the number of the edges joining u and v , and $d(u)$ is the degree of u .

Regarding $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \rightarrow \mathbb{Z}^{|V|}$, its cokernel $\text{coker}(L(G)) = \mathbb{Z}^{|V|}/\text{im}(L(G))$ is an abelian group. For $1 \leq i \leq |V|$, let $e_i = (0, \dots, 0, 1, 0, \dots, 0)^t \in \mathbb{Z}^{|V|}$, be the i -th standard basis, and x_i be its image in $\text{coker}(L(G))$. We know that $\text{coker}(L(G))$ is determined by the generators $x_1, \dots, x_{|V|}$ and the relations

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$(x_1, \dots, x_{|V|})L(G) = 0$. Since $L(G)$ is symmetric, we can rewrite the relations as follows

$$\begin{cases} l_{11}x_1 + l_{12}x_2 + \dots + l_{1|V|}x_{|V|} = 0, \\ l_{21}x_1 + l_{22}x_2 + \dots + l_{2|V|}x_{|V|} = 0, \\ \vdots \\ l_{|V|1}x_1 + l_{|V|2}x_2 + \dots + l_{|V||V|}x_{|V|} = 0. \end{cases} \quad (1.2)$$

Two integral matrices A and B are equivalent (written $A \sim B$) if there are unimodular matrices P and Q such that $B = PAQ$ (An integral matrix P is unimodular if P^{-1} is also integral, i.e., if $\det P = \pm 1$). Equivalently, B is obtainable from A by a sequence of elementary row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by -1 , (3) the addition of any integer times of one row (resp. column) to another row (resp. column).

It is easy to see that $A \sim B$ implies that $\text{coker}(A) \cong \text{coker}(B)$. The Smith normal form is a diagonal canonical form for our equivalence relation: every $n \times n$ integral matrix A is equivalent to a unique diagonal matrix $\text{diag}(s_1(A), \dots, s_n(A))$, where $s_i(A)$ divides $s_{i+1}(A)$ for $i = 1, 2, \dots, n-1$. The i -th diagonal entry of the Smith normal form of A is usually called the i -th invariant factor of A . We will use the fact that the values $s_i(A)$ can also be interpreted as follows: for each i , the product $s_1(A)s_2(A)\dots s_i(A)$ is the greatest common divisor of all $i \times i$ minors of A .

The classification theorem for finitely generated abelian groups asserts that $\text{coker}(L(G))$ has a direct sum decomposition

$$\text{coker}(L(G)) \cong (\mathbb{Z}/t_1\mathbb{Z}) \oplus (\mathbb{Z}/t_2\mathbb{Z}) \oplus \dots \oplus (\mathbb{Z}/t_{|V|}\mathbb{Z}), \quad (1.3)$$

where the nonnegative integers t_i are the diagonal entries of the Smith normal form of the relation matrix $L(G)$, of course, they satisfy that t_i divides t_{i+1} , ($1 \leq i < |V|$). Since G is connected, it is not hard to see that $L(G)$ has rank $|V| - 1$, and the kernel of $L(G)$ is spanned by the vectors in $\mathbb{R}^{|V|}$ which are constant on the vertices. It follows that $t_{|V|} = 0$ and $t_1 \dots t_{|V|-1} \neq 0$.

Now we can write

$$\text{coker}(L(G)) = \mathbb{Z}^{|V|}/\text{im}(L(G)) \cong \mathbb{Z} \oplus K(G), \quad (1.4)$$

where

$$K(G) = (\mathbb{Z}/t_1\mathbb{Z}) \oplus (\mathbb{Z}/t_2\mathbb{Z}) \oplus \dots \oplus (\mathbb{Z}/t_{|V|-1}\mathbb{Z}). \quad (1.5)$$

The finite abelian group $K(G)$ is defined to be the critical group of G . And we will call the positive integers $t_1, \dots, t_{|V|-1}$ the invariant factors of $K(G)$. The critical group $K(G)$ is also known as the Picard group and the Jacobian group of G in [1, 2, 3], while in the physics literature it is known as the abelian sandpile group, and it has a close connection with the critical configuration in a certain dollar game

on G , see [3, 9]. For the general theory of the critical group, we refer the reader to Biggs [2, 3], Godsil [9, Chapter 14], Cori, et al. [5, 6], Dartois et al. [8], and Bacher, et al. [1].

The well known Kirchhoff's Matrix-Tree Theorem [9, Theorem 13.2.1] shows that $t_1 \cdots t_{|V|-1}$ equals the number κ of spanning trees of G . It follows that the invariant factors of $K(G)$ can be used to distinguish pairs of non-isomorphic graphs which have the same κ , and so there is considerable interest in their properties. If G is a simple connected graph, then its Laplacian matrix $L(G)$ has some entry which is equal to -1 . Since the invariant factor t_1 of $K(G)$ is equal to the greatest common divisor of all the entries of $L(G)$, it follows that t_1 must be equal to 1. But the other invariant factors of $K(G)$ are not easy to be determined.

Compared to the number of the results on the spanning tree number κ , there are relatively few results describing the critical group structure of $K(G)$ in terms of the structure of G . Recently, there are some families of graphs for which the critical group structure has been completely determined: wheel graphs [3]; cycles [14]; complete graphs [12]; complete multipartite graphs and cartesian products of complete graphs [11]; a subclass of the threshold graphs [4]; the Möbius ladder graphs [7]; the Cayley graph \mathcal{D}_n of the dihedral group [8]; the square cycle graphs C_n^2 [10]; etc.

Given two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the Cartesian product of them is denoted by $G_1 \times G_2$. It has vertex set $V_1 \times V_2 = \{(u_i, v_j) | u_i \in V_1, v_j \in V_2\}$, where (u_1, v_1) is adjacent to (u_2, v_2) if and only if $u_1 = u_2$ and $(v_1, v_2) \in E_2$, or $(u_1, u_2) \in E_1$ and $v_1 = v_2$. One may view $G_1 \times G_2$ as the graph obtained from G_2 by replacing each of its vertices with a copy of G_1 , and each of its edges with $|V_1|$ edges joining corresponding vertices of G_1 in the two copies.

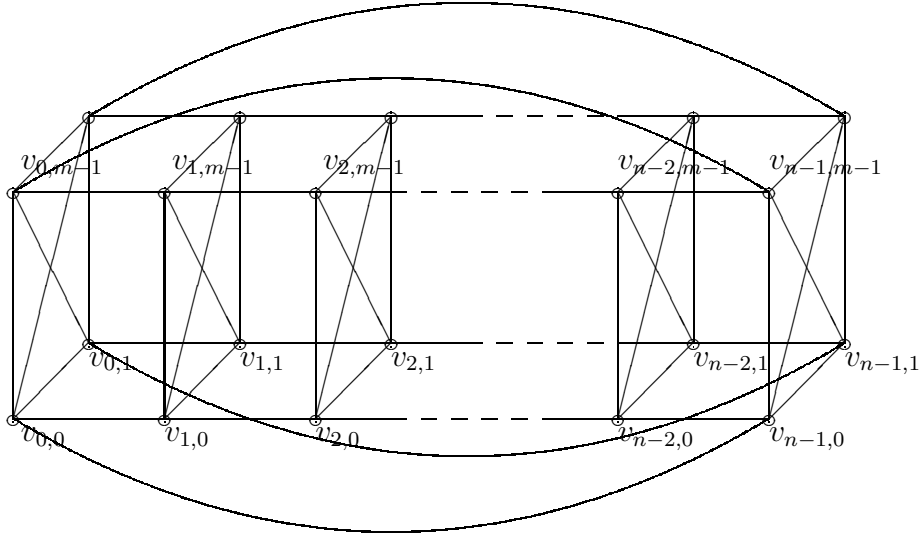


Fig. 1. Graph $K_m \times C_n$.

The structure of the critical group of $K_m \times P_n$ has been obtained in [13], where K_m is the complete graph on m vertices and P_n is the path on n vertices. In this paper we will describe the structure of the critical group on $K_m \times C_n$ with $n, m \geq 3$, where C_n is the cycle on n vertices. From the definition of the Cartesian product of two graphs, it is easy to see that there are n layers of $K_m \times C_n$, each of which is a copy of K_m . Let \mathbb{Z}_n denote $\mathbb{Z}/n\mathbb{Z}$, then for $i \in \mathbb{Z}_n$, $j \in \mathbb{Z}_m$, we may let $v_{i,j}$ denote the j -th vertex in the i -th layer of $K_m \times C_n$. The vertex $v_{i,j}$ is adjacent to vertices $v_{l,j}$ with $l = i \pm 1 \pmod{n}$, and to the vertices $v_{i,k}$, $k \in \mathbb{Z}_m, k \neq j$ (See Fig. 1).

Before the main result can be stated, we need some technical definitions.

If m is a positive integer, let $\alpha = \frac{1}{2}(m+2+\sqrt{m^2+4m})$, $\beta = \frac{1}{2}(m+2-\sqrt{m^2+4m})$. Then for $p \in \mathbb{Z}$, we set $u_p := \frac{1}{\alpha-\beta}(\alpha^p - \beta^p)$, $v_p := \alpha^p + \beta^p$, $\tau_p := \frac{1}{m}(p - u_p)$, $h_p := u_p + u_{p+1}$, and $g_p := \tau_p + \tau_{p+1}$. For the integers a_1, a_2, \dots, a_k , we will let (a_1, a_2, \dots, a_k) denote their greatest common divisor, and use $a_1 \mid a_2 \mid \dots \mid a_k$ to mean that a_1 divides a_2 , a_2 divides a_3 , etc.

Now, we can state our main result in this article as follows.

Theorem 1.1 If $n = 2s + 1$, the critical group of $K_m \times C_n$ ($m, n \geq 3$) is

$$\mathbb{Z}_{(n,g_s)} \oplus \mathbb{Z}_{h_s} \oplus \underbrace{\mathbb{Z}_{h_s} \oplus \dots \oplus \mathbb{Z}_{h_s}}_{m-2} \oplus \mathbb{Z}_\gamma \oplus \underbrace{\mathbb{Z}_{mh_s} \oplus \dots \oplus \mathbb{Z}_{mh_s}}_{m-3} \oplus \mathbb{Z}_\varphi,$$

where $\gamma = \frac{h_s}{(n,g_s)}(n, h_s)$ and $\varphi = \frac{nmh_s}{(n,h_s)}$.

If $n = 2s$, the critical group of $K_m \times C_n$ ($m, n \geq 3$) is

$$\mathbb{Z}_{(u_s, 2\tau_s)} \oplus \mathbb{Z}_\zeta \oplus \underbrace{\mathbb{Z}_{(m,2)u_s} \oplus \dots \oplus \mathbb{Z}_{(m,2)u_s}}_{m-3} \oplus \mathbb{Z}_\eta \oplus \mathbb{Z}_\rho \oplus \underbrace{\mathbb{Z}_\chi \oplus \dots \oplus \mathbb{Z}_\chi}_{m-3} \oplus \mathbb{Z}_\xi,$$

where

$$\begin{cases} \zeta &= \frac{u_s(n, u_s, 4\tau_s)}{(u_s, 2\tau_s)}, \\ \eta &= \frac{u_s(m, 2)(n, u_s - 4\tau_s)}{(n, u_s, 4\tau_s)}, \\ \rho &= \frac{(m+4)u_s(mn, (m+4)u_s, 2n)}{(n, u_s - 4\tau_s)(m, 2)}, \\ \chi &= \frac{m(m+4)u_s}{(m, 2)}, \\ \xi &= \frac{nm(m+4)u_s}{(mn, (m+4)u_s, 2n)}. \end{cases}$$

An immediate consequence of theorem 1.1 is the following Corollary.

Corollary 1.2 The spanning tree number of $K_m \times C_n$ is

$$\frac{n}{m} \left(\left(\frac{m+2+\sqrt{m^2+4m}}{2} \right)^n + \left(\frac{m+2-\sqrt{m^2+4m}}{2} \right)^n - 2 \right)^{m-1}.$$

2 Propositions and Lemmas

We first present some obvious and some less obvious Propositions of the sequences u_p , v_p , τ_p , h_p and g_p .

Note that $\alpha^p \mp \beta^p = (\alpha + \beta)(\alpha^{p-1} \mp \beta^{p-1}) - \alpha\beta(\alpha^{p-2} \mp \beta^{p-2})$. With the above definitions, it is easy to see that $\alpha + \beta = m + 2$ and $\alpha\beta = 1$. So we have the following Proposition 2.1.

Proposition 2.1 If p is integral, then

$$\begin{cases} u_p = (m+2)u_{p-1} - u_{p-2}, \\ u_0 = 0, \quad u_1 = 1, \\ v_p = (m+2)v_{p-1} - v_{p-2}, \\ v_0 = 2, \quad v_1 = m+2. \end{cases} \quad (2.1)$$

From (2.1), it is easy to see that for every integer p , u_p and v_p are integral. The following Propositions 2.2 can be proved by induction on p .

Proposition 2.2 If p is integral, then

$$u_p \equiv p \pmod{m}, \quad v_p \equiv 2 \pmod{m}. \quad (2.2)$$

By (2.2), we see that $m \mid (p - u_p)$, i.e., τ_p is integral for $p \in \mathbb{Z}$. In fact, we further have the following Proposition (2.3).

Proposition 2.3 If p is integral, then

$$\tau_p = (m+2)\tau_{p-1} - \tau_{p-2} - (p-1). \quad (2.3)$$

Proof Since $u_p = p - m\tau_p$, it follows from Proposition 2.1 that $p - m\tau_p = (m+2)(p-1 - m\tau_{p-1}) - (p-2 - m\tau_{p-2})$. So $m\tau_p = m(m+2)\tau_{p-1} - m\tau_{p-2} - m(p-1)$ and then (2.3) holds. \square

Proposition 2.4 If p is a nonnegative integer, then

$$\begin{aligned} & u_{p-1}u_{p+1} - u_p^2 - 1 + (u_{p+1} - u_{p-1}) \\ &= v_p - 2 = \begin{cases} mh_s^2, & \text{if } p = 2s + 1, \\ m(m+4)u_s^2, & \text{if } p = 2s. \end{cases} \end{aligned} \quad (2.4)$$

Proof A direct calculation can show

$$\begin{aligned} & u_{p-1}u_{p+1} - u_p^2 - 1 + (u_{p+1} - u_{p-1}) \\ &= \frac{\alpha^{p-1} - \beta^{p-1}}{\alpha - \beta} \cdot \frac{\alpha^{p+1} - \beta^{p+1}}{\alpha - \beta} - \left(\frac{\alpha^p - \beta^p}{\alpha - \beta} \right)^2 - 1 + \left(\frac{\alpha^{p+1} - \beta^{p+1}}{\alpha - \beta} - \frac{\alpha^{p-1} - \beta^{p-1}}{\alpha - \beta} \right) \\ &= -1 - 1 + \alpha^p + \beta^p = v_p - 2. \end{aligned}$$

So the first equality of (2.4) holds. Now we verify the second equality.

If $p = 2s + 1$, then

$$\begin{aligned} mh_s^2 &= m(u_{s+1} + u_s)^2 = \frac{m}{(\alpha - \beta)^2} (\alpha^{s+1} - \beta^{s+1} + \alpha^s - \beta^s)^2 \\ &= \frac{m}{m^2 + 4m} (\alpha^{2s+2} + \beta^{2s+2} + \alpha^{2s} + \beta^{2s} + 2\alpha^{2s+1} + 2\beta^{2s+1} - 2 - 2 - 2\alpha - 2\beta) \\ &= \frac{1}{m+4} ((m+4)v_p - 2(m+4)) = v_p - 2. \end{aligned}$$

If $p = 2s$, then

$$m(m+4)u_s^2 = \frac{m(m+4)}{(\alpha-\beta)^2}(\alpha^s - \beta^s)^2 = \alpha^{2s} + \beta^{2s} - 2 = v_p - 2.$$

□

Proposition 2.5 If p is integral, then

$$(u_{p+1} - 1, u_p) = (u_p, u_{p-1} + 1) = \begin{cases} h_s, & \text{if } p = 2s + 1, \\ (m, 2)u_s, & \text{if } p = 2s. \end{cases} \quad (2.5)$$

Proof For $i \in \mathbb{Z}$, set $\theta_i := u_{p-i} + u_i$. Note that $\alpha\beta = 1$ implies that $u_{-i} = -u_i$, then it follows from (2.1) that

$$\begin{aligned} \theta_{i+1} &= u_{p-i-1} + u_{i+1} = -u_{i+1-p} + u_{i+1} \\ &= -((m+2)u_{i-p} - u_{i-1-p}) + (m+2)u_i - u_{i-1} \\ &= (m+2)u_{p-i} + u_{i-1-p} + (m+2)u_i - u_{i-1} \\ &= (m+2)(u_{p-i} + u_i) - (u_{p-(i-1)} + u_{i-1}) \\ &= (m+2)\theta_i - \theta_{i-1}. \end{aligned} \quad (2.6)$$

Here we are using the fact that $(a, b) = (a, ax - b)$ for $a, b, x \in \mathbb{Z}$. Thus

$$\begin{aligned} (u_{p+1} - 1, u_p) &= (\theta_{-1}, \theta_0) \\ &= ((m+2)\theta_0 - \theta_{-1}, \theta_0) = (\theta_1, \theta_0) \\ &= (u_p, u_{p-1} + 1). \end{aligned}$$

Moreover $(\theta_0, \theta_1) = (\theta_1, (m+2)\theta_1 - \theta_0) = (\theta_1, \theta_2) = \dots = (\theta_{s-1}, \theta_s)$, where $s = \lfloor \frac{p}{2} \rfloor$.

If $p = 2s + 1$, then $\theta_{s-1} = u_{s+2} + u_{s-1} = (m+1)h_s$ and $\theta_s = u_{s+1} + u_s = h_s$. Thus

$$(\theta_{s-1}, \theta_s) = h_s.$$

If $p = 2s$, then $\theta_{s-1} = u_{s+1} + u_{s-1} = (m+2)u_s$ and $\theta_s = 2u_s$. Therefore

$$(\theta_{s-1}, \theta_s) = ((m+2)u_s, 2u_s) = (m, 2)u_s.$$

□

The following Lemmas 2.6 and 2.7 will be used in the proof of Theorem 1.1.

Lemma 2.6 For $n \in \mathbb{N}$, let $B = \begin{pmatrix} n & \tau_{n-1} & \tau_n \\ 0 & \tau_n - \tau_{n-1} & \tau_{n+1} - \tau_n \\ 0 & u_n & u_{n+1} - 1 \end{pmatrix}$ and $\text{diag}(s_1(B), s_2(B), s_3(B))$

its Smith normal form.

If $n = 2s + 1$, then

$$\begin{cases} s_1(B) = (n, g_s), \\ s_2(B) = h_s, \\ s_3(B) = \frac{nh_s}{(n, g_s)}. \end{cases} \quad (2.7)$$

If $n = 2s$, then

$$\begin{cases} s_1(B) = (u_s, 2\tau_s), \\ s_2(B) = \frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}, \\ s_3(B) = \frac{n(m+4)u_s}{(n, u_s - 4\tau_s)}. \end{cases} \quad (2.8)$$

Proof Recall that $s_1(B)$ equals the greatest common divisor of all entries of B . So

$$\begin{aligned} s_1(B) &= (n, \tau_{n-1}, \tau_n, \tau_n - \tau_{n-1}, \tau_{n+1} - \tau_n, u_n, u_{n+1} - 1) \\ &= (n, \tau_{n-1}, \tau_n, \tau_{n+1}, u_n, u_{n+1} - 1). \end{aligned}$$

Since we have (2.3) and

$$\begin{cases} u_n = n - m\tau_n, \\ u_{n+1} - 1 = (m+1)n + m\tau_{n-1} - m(m+2)\tau_n, \end{cases}$$

it follows that

$$s_1(B) = (n, \tau_n, \tau_{n-1}) = \left(n, \frac{1}{m}(n - \theta_0), \frac{1}{m}(n - \theta_1) \right). \quad (2.9)$$

From (2.6), we have $\frac{1}{m}(n - \theta_2) = (m+2)\frac{1}{m}(n - \theta_1) - \frac{1}{m}(n - \theta_0) - n$. Therefore

$$\begin{aligned} s_1(B) &= \left(n, \frac{1}{m}(n - \theta_0), \frac{1}{m}(n - \theta_1) \right) \\ &= \left(n, \frac{1}{m}(n - \theta_1), \frac{1}{m}(n - \theta_2) \right) \\ &= \dots \\ &= \left(n, \frac{1}{m}(n - \theta_{s-1}), \frac{1}{m}(n - \theta_s) \right), \end{aligned}$$

where $s = \lfloor \frac{n}{2} \rfloor$.

• If $n = 2s + 1$, then $\theta_{s-1} = u_{s+2} + u_{s-1} = (m+1)h_s$ and $\theta_s = u_{s+1} + u_s = h_s$. It results that

$$s_1(B) = \left(n, \frac{1}{m}(n - (m+1)h_s), \frac{1}{m}(n - h_s) \right) = (n, g_s). \quad (2.10)$$

• If $n = 2s$, then $\theta_{s-1} = u_{s+1} + u_{s-1} = (m+2)u_s$ and $\theta_s = 2u_s$. It results that

$$s_1(B) = \left(n, \frac{1}{m}(n - (m+2)u_s), \frac{1}{m}(n - 2u_s) \right) = (2s, 2\tau_s, u_s) = (u_s, 2\tau_s). \quad (2.11)$$

Recall that $s_1(B)s_2(B)$ equals the greatest common divisor of all 2×2 minors of B . So

$$s_1(B)s_2(B) = (\Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{22}, \Delta_{23}, \Delta_{31}, \Delta_{32}, \Delta_{33}),$$

where Δ_{ij} is the determinant of the submatrix formed by deleting the i -th row and j -th column of the matrix B for $1 \leq i, j \leq 3$. It is straightforward to see that

$$\begin{aligned}
\Delta_{11} &= \det \begin{pmatrix} \tau_n - \tau_{n-1} & \tau_{n+1} - \tau_n \\ u_n & u_{n+1} - 1 \end{pmatrix} = \frac{1}{m}(u_{n-1}u_{n+1} - u_n^2 - 1 + (u_{n+1} - u_{n-1})) \stackrel{(2.4)}{=} \\
&\frac{1}{m}(v_n - 2); \Delta_{12} = \det \begin{pmatrix} 0 & \tau_{n+1} - \tau_n \\ 0 & u_{n+1} - 1 \end{pmatrix} = 0; \Delta_{13} = \det \begin{pmatrix} 0 & \tau_n - \tau_{n-1} \\ 0 & u_n \end{pmatrix} = 0; \Delta_{21} = \\
&\det \begin{pmatrix} \tau_{n-1} & \tau_n \\ u_n & u_{n+1} - 1 \end{pmatrix} = \frac{1}{m}((n-1-u_{n-1})(u_{n+1}-1) - (n-u_n)u_n) = \frac{1}{m}(n(u_{n+1} - \\
&1 - u_n)) - \frac{1}{m}(u_{n-1}u_{n+1} - u_n^2 - 1 + (u_{n+1} - u_{n-1})) \stackrel{(2.4)}{=} n(\tau_n - \tau_{n+1}) - \frac{1}{m}(v_n - 2); \\
\Delta_{22} &= \det \begin{pmatrix} n & \tau_n \\ 0 & u_{n+1} - 1 \end{pmatrix} = n(u_{n+1} - 1); \Delta_{23} = \det \begin{pmatrix} n & \tau_{n-1} \\ 0 & u_n \end{pmatrix} = nu_n; \Delta_{31} = \\
&\det \begin{pmatrix} \tau_{n-1} & \tau_n \\ \tau_n - \tau_{n-1} & \tau_{n+1} - \tau_n \end{pmatrix} = \frac{1}{m^2}((n-1-u_{n-1})(u_n - u_{n+1} + 1) - (n-u_n)(u_{n-1} + 1 - \\
&u_n)) = \frac{1}{m^2}((u_{n-1}u_{n+1} - u_n^2 - 1 + (-u_{n-1} + u_{n+1})) + 2nu_n - n(u_{n+1} + u_{n-1})) \stackrel{(2.4)}{=} \frac{1}{m}(\frac{v_n-2}{m} - \\
&nu_n); \Delta_{32} = \det \begin{pmatrix} n & \tau_n \\ 0 & \tau_{n+1} - \tau_n \end{pmatrix} = n(\tau_{n+1} - \tau_n); \Delta_{33} = \det \begin{pmatrix} n & \tau_{n-1} \\ 0 & \tau_n - \tau_{n-1} \end{pmatrix} = \\
&n(\tau_n - \tau_{n-1}).
\end{aligned}$$

Note that $\Delta_{33} = \Delta_{23} + \Delta_{32}$, $\Delta_{21} = -\Delta_{32} - \Delta_{11}$ and $\Delta_{11} = m\Delta_{31} + \Delta_{23}$. So

$$\begin{aligned}
s_1(B)s_2(B) &= (\Delta_{22}, \Delta_{23}, \Delta_{31}, \Delta_{32}) \\
&= (n(u_{n+1} - 1), nu_n, \frac{1}{m}(\frac{v_n-2}{m} - nu_n), n(\tau_{n+1} - \tau_n)) \\
&\stackrel{\theta_i=u_{n-i}+u_i}{=} (n\theta_{-1}, n\theta_0, \frac{1}{m}(\frac{v_n-2}{m} - n\theta_0), \frac{n}{m}(\theta_0 - \theta_{-1})).
\end{aligned} \tag{2.12}$$

With the aid of (2.6), it is easy to verify that $\frac{n}{m}(\theta_0 - \theta_{-1}) + n\theta_0 = \frac{n}{m}(\theta_1 - \theta_0)$. Moreover, we have $\frac{1}{m}(\frac{v_n-2}{m} - n\theta_0) - \frac{n}{m}(\theta_1 - \theta_0) = \frac{1}{m}(\frac{v_n-2}{m} - n\theta_1)$. From (2.12), it follows that

$$\begin{aligned}
s_1(B)s_2(B) &= (n\theta_0, n\theta_1, \frac{n}{m}(\theta_1 - \theta_0), \frac{1}{m}(\frac{v_n-2}{m} - n\theta_1)) \\
&= (n\theta_1, n\theta_2, \frac{n}{m}(\theta_2 - \theta_1), \frac{1}{m}(\frac{v_n-2}{m} - n\theta_2)) \\
&= \dots \\
&= (n\theta_{s-1}, n\theta_s, \frac{n}{m}(\theta_s - \theta_{s-1}), \frac{1}{m}(\frac{v_n-2}{m} - n\theta_s)),
\end{aligned} \tag{2.13}$$

where $s = \lfloor \frac{n}{2} \rfloor$.

• If $n = 2s + 1$, then $\theta_{s-1} = u_{s+2} + u_{s-1} = (m+1)h_s$ and $\theta_s = h_s$. So from (2.4) and (2.13), we can see that

$$\begin{aligned}
s_1(B)s_2(B) &= (n(m+1)h_s, nh_s, nh_s, \frac{1}{m}(h_s^2 - nh_s)) \\
&= (nh_s, \frac{1}{m}(h_s^2 - nh_s)) \\
&= h_s(n, \frac{1}{m}(h_s - n)) = h_s(n, g_s).
\end{aligned} \tag{2.14}$$

• If $n = 2s$, then $\theta_{s-1} = u_{s+1} + u_{s-1} = (m+2)u_s$, and $\theta_s = 2u_s$. So from (2.4) and (2.13), we can see that

$$\begin{aligned}
s_1(B)s_2(B) &= (n\theta_{s-1}, n\theta_s, \frac{n}{m}(\theta_s - \theta_{s-1}), \frac{1}{m}((m+4)u_s^2 - 2nu_s)) \\
&= (nu_s, \frac{1}{m}((m+4)u_s^2 - 2nu_s)) \\
&= u_s(n, \frac{1}{m}((m+4)u_s - 2n)) \\
&= u_s(n, u_s - 4\tau_s).
\end{aligned} \tag{2.15}$$

Recall that $s_1(B)s_2(B)s_3(B)$ equals the determinant of B . So

$$s_1(B)s_2(B)s_3(B) = \det(B) = n\Delta_{11}(B) = \frac{n}{m}(v_n - 2).$$

Thus we have

$$\begin{cases} s_1(B)s_2(B)s_3(B) = nh_s^2, & \text{if } n = 2s + 1, \\ s_1(B)s_2(B)s_3(B) = n(m + 4)u_s^2, & \text{if } n = 2s. \end{cases} \quad (2.16)$$

Combining (2.10), (2.11), (2.14), (2.15), and (2.16), we obtain the formulas (2.7) and (2.8). \square

Lemma 2.7 For $n \in \mathbb{N}$, let $W = \begin{pmatrix} u_{n-1} + 1 & u_n \\ u_n & u_{n+1} - 1 \end{pmatrix}$, and $\text{diag}(s_1(W), s_2(W))$ its Smith normal form.

If $n = 2s + 1$, then

$$\begin{cases} s_1(W) = h_s, \\ s_2(W) = mh_s. \end{cases}$$

If $n = 2s$, then

$$\begin{cases} s_1(W) = (m, 2)u_s, \\ s_2(W) = \frac{m(m+4)u_s}{(m, 2)}. \end{cases}$$

Proof Recall that $s_1(W)$ equals the greatest common divisor of all the entries of W . So

$$\begin{aligned} s_1(W) &= (u_{n-1} + 1, u_n, u_{n+1} - 1) = (u_{n-1} + 1, u_n) \\ &\stackrel{(2.5)}{=} \begin{cases} s_1(W) = h_s, & \text{if } n = 2s + 1, \\ s_1(W) = (m, 2)u_s, & \text{if } n = 2s. \end{cases} \end{aligned} \quad (2.17)$$

Recall that $s_1(W)s_2(W)$ equals the greatest common divisor of all 2×2 minors of W . So

$$\begin{aligned} s_1(W)s_2(W) &= \det(W) = u_{n-1}u_{n+1} - u_n^2 - 1 + (u_{n+1} - u_{n-1}) \\ &\stackrel{(2.4)}{=} \begin{cases} s_1(W)s_2(W) = mh_s^2, & \text{if } n = 2s + 1, \\ s_1(W)s_2(W) = m(m + 4)u_s^2, & \text{if } n = 2s. \end{cases} \end{aligned} \quad (2.18)$$

Combining (2.17) and (2.18), we can obtain

$$s_2(W) = \begin{cases} mh_s, & \text{if } n = 2s + 1, \\ \frac{m(m+4)u_s}{(m, 2)}, & \text{if } n = 2s. \end{cases} \quad (2.19)$$

\square

3 Proofs of Theorem 1.1 and Corollary 1.2

Observe that the critical group of graph G is completely determined by the cokernel of $L(G)$. Thus, it is sufficient to compute the Smith normal form of the Laplacian matrix $L(G)$.

The proof of Theorem 1.1 contains the following steps:

- (1) First, we prove that there is a matrix $A \in \mathbb{Z}^{2m \times 2m}$ such that $L(G) \sim I_{nm-2m} \oplus A$, (see (3.9) and (3.10)).
- (2) Next, we prove there are two matrices $B \in \mathbb{Z}^{3 \times 3}$ and $W \in \mathbb{Z}^{2 \times 2}$ such that $A \sim 0_1 \oplus B \oplus \underbrace{W \oplus \cdots \oplus W}_{m-2}$; the Smith normal forms of B and W are given in Lemmas 2.6 and 2.7 respectively.
- (3) Finally, we compute the Smith normal form of A from those of B and W .

After the three steps, the Smith normal form of $L(G)$ will be obtained.

Step 1

Now we work on the system of relations of the cokernel of the Laplacian of $K_m \times C_n$. Let $e_{i,j} = (0, \dots, 0, 1, 0, \dots, 0)^t \in \mathbb{Z}^{mn}$, whose unique nonzero 1 is in the position corresponding to vertex $v_{i,j}$, and let $x_{i,j}$ be its image in $\text{coker}(L(K_m \times C_n))$. Then it follows from the relations (1.2) of $\text{coker}(L(K_m \times C_n))$ that we can get the system of equations:

$$(m+1)x_{i,j} - \sum_{\substack{k \in \mathbb{Z}_m \\ k \neq j}} x_{i,k} - x_{i-1,j} - x_{i+1,j} = 0, \quad i \in \mathbb{Z}_n, j \in \mathbb{Z}_m. \quad (3.1)$$

Let $M_i = \sum_{j \in \mathbb{Z}_m} x_{i,j}$, for $i \in \mathbb{Z}_n$. Then from (3.1) we have

$$(m+1)M_i - (m-1)M_i - M_{i+1} - M_{i-1} = 0. \quad (3.2)$$

This identity implies that

$$M_{i+1} = 2M_i - M_{i-1}. \quad (3.3)$$

Recursively using identity (3.3), we can rewrite all M_i 's as integral linear combinations of M_0 and M_1 .

$$M_i = iM_1 - (i-1)M_0, \quad 2 \leq i \leq n-1. \quad (3.4)$$

So from (3.1) and (3.4), we have

$$x_{i,j} = (m+2)x_{i-1,j} - x_{i-2,j} + (i-2)M_0 - (i-1)M_1, \quad (3.5)$$

where $2 \leq i \leq n-1$, $0 \leq j \leq m-1$.

Lemma 3.1 For $0 \leq i \leq n-1$, $0 \leq j \leq m-1$, we have

$$x_{i,j} = -u_{i-1}x_{0,j} + u_i x_{1,j} - \tau_{i-1}M_0 + \tau_i M_1. \quad (3.6)$$

Proof This Lemma is valid in cases $i = 0, 1, 2$. Suppose that $x_{l,j} = -u_{l-1}x_{0,j} + u_l x_{1,j} - \tau_{l-1}M_0 + \tau_l M_1$, for $l \leq h-1$, where $h \geq 3$. Then from the induction assumption and the equations (3.5), it follows that

$$\begin{aligned} x_{h,j} &= (m+2)x_{h-1,j} - x_{h-2,j} + (h-2)M_0 - (h-1)M_1, \\ &= (m+2)(-u_{h-2}x_{0,j} + u_{h-1}x_{1,j} - \tau_{h-2}M_0 + \tau_{h-1}M_1) \\ &\quad - (-u_{h-3}x_{0,j} + u_{h-2}x_{1,j} - \tau_{h-3}M_0 + \tau_{h-2}M_1) + (h-2)M_0 - (h-1)M_1 \\ &= (-(m+2)u_{h-2} + u_{h-3})x_{0,j} + ((m+2)u_{h-1} - u_{h-2})x_{1,j} \\ &\quad + (-(m+2)\tau_{h-2} + \tau_{h-3} + (h-2))M_0 + ((m+2)\tau_{h-1} - \tau_{h-2} - (h-1))M_1 \\ &= -u_{h-1}x_{0,j} + u_h x_{1,j} - \tau_{h-1}M_0 + \tau_h M_1. \end{aligned}$$

Recall (2.1) and (2.3), we know that (3.6) holds by induction. \square

In view of Lemma 3.1, we only need at most $2m$ generators for the system of equations (3.1). Indeed each $x_{i,j}$ can be expressed in terms of $x_{0,0}, x_{1,0}, x_{0,1}, x_{1,1}, \dots, x_{0,m-1}, x_{1,m-1}$. So we know that there are at least $nm - 2m$ diagonal entries of the Smith normal form of $L(G)$ are equal to 1 and the remaining invariant factors of the abelian group $\text{coker} L(K_m \times C_n)$ are the diagonal entries of the Smith normal form of the relations matrix induced by $x_{0,0}, x_{1,0}, x_{0,1}, x_{1,1}, \dots, x_{0,m-1}, x_{1,m-1}$.

From (3.6) and the cyclic structure of $K_m \times C_n$, it follows that, for $0 \leq j \leq m-1$,

$$\begin{cases} x_{0,j} = x_{n,j} = -u_{n-1}x_{0,j} + u_n x_{1,j} - \tau_{n-1}M_0 + \tau_n M_1, \\ x_{1,j} = x_{n+1,j} = -u_n x_{0,j} + u_{n+1}x_{1,j} - \tau_n M_0 + \tau_{n+1}M_1. \end{cases} \quad (3.7)$$

Therefore, for $0 \leq j \leq m-1$,

$$\begin{cases} (-u_{n-1} - \tau_{n-1} - 1)x_{0,j} + (u_n + \tau_n)x_{1,j} - \tau_{n-1} \sum_{k \neq j} x_{0,k} + \tau_n \sum_{k \neq j} x_{1,k} = 0, \\ (-u_n - \tau_n)x_{0,j} + (u_{n+1} + \tau_{n+1} - 1)x_{1,j} - \tau_n \sum_{k \neq j} x_{0,k} + \tau_{n+1} \sum_{k \neq j} x_{1,k} = 0. \end{cases} \quad (3.8)$$

Let

$$E = \begin{pmatrix} -u_{n-1} - 1 - \tau_{n-1} & u_n + \tau_n \\ -u_n - \tau_n & u_{n+1} - 1 + \tau_{n+1} \end{pmatrix}, \quad F = \begin{pmatrix} -\tau_{n-1} & \tau_n \\ -\tau_n & \tau_{n+1} \end{pmatrix},$$

and

$$Y = (x_{0,0}, x_{1,0}, x_{0,1}, x_{1,1}, \dots, x_{0,m-1}, x_{1,m-1})^T.$$

Then from the equalities in (3.8), we have that

$$AY = 0, \quad (3.9)$$

where

$$A = \begin{pmatrix} E & F & F & \cdots & F \\ F & E & F & \cdots & F \\ F & F & E & \cdots & F \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F & F & F & \cdots & E \end{pmatrix} \in \mathbb{Z}^{2m \times 2m}. \quad (3.10)$$

Step 2

The matrix A in equation (3.9) is the relation matrix induced by the generators $x_{0,0}, x_{1,0}, x_{0,1}, x_{1,1}, \dots, x_{0,m-1}, x_{1,m-1}$. We now discuss the Smith normal form of the relation matrix A .

Let

$$H = \begin{pmatrix} I_2 & 0 & 0 & \cdots & 0 \\ -(m-1)I_2 & I_2 & I_2 & \cdots & I_2 \\ -I_2 & 0 & I_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -I_2 & 0 & 0 & \cdots & I_2 \end{pmatrix} \in \mathbb{Z}^{2m \times 2m},$$

where I_2 is the 2×2 identity matrix. Then it is not difficult to verify that

$$H^{-1} = \begin{pmatrix} I_2 & 0 & 0 & \cdots & 0 \\ I_2 & I_2 & -I_2 & \cdots & -I_2 \\ I_2 & 0 & I_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_2 & 0 & 0 & \cdots & I_2 \end{pmatrix} \in \mathbb{Z}^{2m \times 2m}.$$

By a direct calculation, we have

$$HAH^{-1} = \begin{pmatrix} E + (m-1)F & F & 0 & 0 & 0 \\ 0 & E - F & 0 & 0 & 0 \\ 0 & 0 & E - F & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & E - F \end{pmatrix}. \quad (3.11)$$

Note that

$$\begin{pmatrix} E + (m-1)F & F \\ 0 & E - F \end{pmatrix} = \begin{pmatrix} -n & n & -\tau_{n-1} & \tau_n \\ -n & n & -\tau_n & \tau_{n+1} \\ 0 & 0 & -u_{n-1} - 1 & u_n \\ 0 & 0 & -u_n & u_{n+1} - 1 \end{pmatrix}.$$

$$\text{Let } Q_1 = \begin{pmatrix} m & -m & 1 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } Q_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \text{ Then it is easy}$$

to see that Q_1 and Q_2 are unimodular matrices and a careful calculation can show

$$Q_1 \begin{pmatrix} E + (m-1)F & F \\ 0 & E - F \end{pmatrix} Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \quad (3.12)$$

where the matrix B is just the one defined in Lemma 2.6.

Note that $(E - F) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u_{n-1} + 1 & u_n \\ u_n & u_{n+1} - 1 \end{pmatrix} = W$, which is just the one considered in Lemma 2.7. Therefore, from (3.11) and (3.12) we have

$$\begin{aligned} A &\sim HAH^{-1} \sim 0_1 \oplus B \oplus \underbrace{W \oplus \cdots \oplus W}_{m-2} \\ &\sim \text{diag}(s_1(B), s_2(B), s_3(B), s_1(W), \dots, s_1(W), s_2(W), \dots, s_2(W), 0) \\ &\sim \text{diag}(s_1(B), s_2(B), s_1(W), \dots, s_1(W), s_2(W), \dots, s_2(W), s_3(B), 0). \end{aligned} \quad (3.13)$$

Step 3

Now we distinguish two cases to compute the Smith normal form of A .

Case 1. $n = 2s + 1$.

Then by Lemmas 2.6 and 2.7 we have that $s_1(B) = (n, g_s)$, $s_2(B) = h_s$, $s_3(B) = \frac{nh_s}{(n, g_s)}$, $s_1(W) = h_s$, $s_2(W) = mh_s$. Since $s_1(B) | s_2(B)$, $s_1(W) | s_2(W)$ and $s_2(B) = s_1(W)$, it follows that $s_1(B) | s_2(W)$.

Write $\gamma = (s_2(W), s_3(B))$ and $\varphi = \frac{s_2(W)s_3(B)}{\gamma}$. Then

$$\text{diag}(s_2(W), s_3(B)) \sim \text{diag}(\gamma, \varphi), \quad (3.14)$$

where

$$\begin{aligned} \gamma &= \left(mh_s, \frac{nh_s}{(n, g_s)} \right) = \frac{h_s}{(n, g_s)} (m(n, g_s), n) \\ &= \frac{h_s}{(n, g_s)} (mn, n - h_s, n) \quad (\text{Here } mg_s = n - h_s.) \\ &= \frac{h_s}{(n, g_s)} (n, h_s), \end{aligned} \quad (3.15)$$

and

$$\varphi = \frac{nmh_s}{(n, h_s)}. \quad (3.16)$$

Note that $s_1(W) = s_2(B)$, it implies that $s_1(W)|s_3(B)$. So $s_1(W)|\gamma|s_2(W)$. Moreover, it is easy to see that $s_2(W) \mid \varphi$.

Therefore, (3.13) and (1.14) implies that

$$\text{diag}(s_1(B), s_2(B), \underbrace{s_1(W), \dots, s_1(W)}_{m-2}, \gamma, \underbrace{s_2(W), \dots, s_2(W)}_{m-3}, \varphi, 0) \quad (3.17)$$

is the Smith normal form of A .

Case 2. $n = 2s$.

Then by Lemmas 2.6 and 2.7, we know that in this case $s_1(B) = (u_s, 2\tau_s)$, $s_2(B) = \frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}$, $s_3(B) = \frac{n(m+4)u_s}{(n, u_s - 4\tau_s)}$, $s_1(W) = (m, 2)u_s$, $s_2(W) = \frac{m(m+4)u_s}{(m, 2)}$.

It is obvious that

$$s_1(B)|s_2(B)|s_3(B), \quad s_1(B)|s_1(W)|s_2(W), \quad s_2(B)|s_2(W). \quad (3.18)$$

Clearly, we have

$$\text{diag}(s_2(W), s_3(B)) \sim \text{diag}(\rho, \xi), \quad (3.19)$$

where

$$\begin{aligned} \rho &= (s_2(W), s_3(B)) = \left(\frac{m(m+4)u_s}{(m, 2)}, \frac{n(m+4)u_s}{(n, u_s - 4\tau_s)} \right) \\ &= \frac{(m+4)u_s}{(n, u_s - 4\tau_s)(m, 2)} (mn, (m+4)u_s - 2n, mn, 2n) \\ &= \frac{(m+4)u_s(mn, (m+4)u_s, 2n)}{(n, u_s - 4\tau_s)(m, 2)}, \end{aligned} \quad (3.20)$$

and

$$\xi = \frac{s_2(W)s_3(B)}{\rho} = \frac{mn(m+4)u_s}{(mn, (m+4)u_s, 2n)}. \quad (3.21)$$

Therefore, It follows from (3.13) that

$$A \sim \text{diag}(s_1(B), s_2(B), \underbrace{s_1(W), \dots, s_1(W)}_{m-2}, \rho, \underbrace{s_2(W), \dots, s_2(W)}_{m-3}, \xi, 0). \quad (3.22)$$

We also have

$$\text{diag}(s_2(B), s_1(W)) \sim \text{diag}(\zeta, \eta), \quad (3.23)$$

where

$$\begin{aligned} \zeta &= (s_2(B), s_1(W)) = \left(\frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}, (m, 2)u_s \right) \\ &= \frac{u_s(n, u_s - 4\tau_s, mu_s, 2m\tau_s, 2u_s, 4\tau_s)}{(u_s, 2\tau_s)} \\ &= \frac{u_s(n, u_s, 4\tau_s)}{(u_s, 2\tau_s)}, \end{aligned} \quad (3.24)$$

and

$$\eta = \frac{s_2(B)s_1(W)}{\zeta} = \frac{u_s(m, 2)(n, u_s - 4\tau_s)}{(n, u_s, 4\tau_s)}. \quad (3.25)$$

Combining (3.22) and (3.23), we have

$$A \sim \text{diag}(s_1(B), \zeta, \underbrace{s_1(W), \dots, s_1(W)}_{m-3}, \eta, \rho, \underbrace{s_2(W), \dots, s_2(W)}_{m-3}, \xi, 0).$$

Since $(n, u_s - 4\tau_s) | n$, and $(m, 2) u_s | (m+4)u_s$, it follows that $s_1(W) | s_3(B)$ and hence $s_1(W) | (s_2(W), s_3(B))$. Furthermore, since $s_2(B) | s_3(B)$ and $s_2(B) | s_2(W)$, it implies that $s_2(B) | (s_2(W), s_3(B))$. So, $(s_2(W), s_3(B))$, i.e., ρ is a common multiple of $s_1(W)$ and $s_2(B)$. Note that η is the least common multiple of $s_1(W)$ and $s_2(B)$, it divides ρ . According to (3.18), it is easy to see $s_1(B) | \zeta$. And it is clear that we have $s_1(W) | \eta$ and $s_2(W) | \xi$. Thus

$$\text{diag}(s_1(B), \zeta, \underbrace{s_1(W), \dots, s_1(W)}_{m-3}, \eta, \rho, \underbrace{s_2(W), \dots, s_2(W)}_{m-3}, \xi, 0) \quad (3.26)$$

is the Smith normal form of A .

Now, the proof of Theorem 1.1 is completed. \square

Proof of Corollary 1.2 If $n = 2s + 1$, then $s_1(W)s_2(W) = mh_s^2 \stackrel{(2.4)}{=} v_n - 2$ and $s_1(B)s_2(B)s_1(W)\gamma\varphi = (n, g_s)h_s h_s \frac{h_s}{(n, g_s)}(n, h_s) \frac{nmh_s}{(n, h_s)} = nmh_s^4 \stackrel{(2.4)}{=} \frac{n}{m}(v_n - 2)^2$. It follows that the spanning tree number of $K_m \times C_n$ is $\frac{n}{m}(v_n - 2)^2 \times (v_n - 2)^{m-3} = \frac{n}{m}(v_n - 2)^{m-1}$.

If $n = 2s$, then $s_1(W)s_2(W) = (m, 2)u_s \frac{m(m+4)u_s}{(m, 2)} = m(m+4)u_s^2 \stackrel{(2.4)}{=} v_n - 2$ and $s_1(B)\zeta\eta\rho\xi = nm(m+4)u_s^4 \stackrel{(2.4)}{=} \frac{n}{m}(v_n - 2)^2$. It follows that the spanning tree number of $K_m \times C_n$ is $\frac{n}{m}(v_n - 2)^2 \times (v_n - 2)^{m-3} = \frac{n}{m}(v_n - 2)^{m-1}$.

(In fact, by (3.13) we know that the spanning tree number of graph $K_m \times C_n$ is $|\det(\text{diag}(B \oplus \underbrace{W \oplus \dots \oplus W}_{m-2}))| = \det(B) \times (\det(W))^{m-2}$. In the proofs of Lemmas

2.6 and 2.7, we have seen that $\det(B) = \frac{n}{m}(v_n - 2)$ and $\det(W) = v_n - 2$. So the spanning tree number is $\frac{n}{m}(v_n - 2)^{m-1}$.) \square

4 Remarks

(I) If $n = 1$, then $K_m \times C_n$ is the complete graph K_m , from [12] we know its critical group is $(\mathbb{Z}_m)^{m-2}$.

(II) If $n = 2$, then $\text{coker}(K_m \times C_2)$ is determined by the generators $x_{0,j}, x_{1,j}$, and the relations

$$\begin{cases} (m+1)x_{0,j} - \sum_{\substack{k \in \mathbb{Z}_m \\ k \neq j}} x_{0,k} - 2x_{1,j} = 0, \\ (m+1)x_{1,j} - \sum_{\substack{k \in \mathbb{Z}_m \\ k \neq j}} x_{1,k} - 2x_{0,j} = 0, \end{cases} \quad (4.1)$$

$$\quad (4.2)$$

Where $j \in \mathbb{Z}_m$. From (4.1), we get

$$2x_{1,j} = (m+1)x_{0,j} - \sum_{\substack{k \in \mathbb{Z}_m \\ k \neq j}} x_{0,k}, \quad j \in \mathbb{Z}_m. \quad (4.3)$$

Substituting (4.3) into (4.2) $\times 2$ gives the following

$$(m+4)(m-1)x_{0,j} - \sum_{\substack{k \in \mathbb{Z}_m \\ k \neq j}} (m+4)x_{0,k} = 0, \quad j \in \mathbb{Z}_m.$$

So we can simplify (up to equivalence) the Laplacian matrix of $K_m \times C_2$ into

$$(m+4) \begin{pmatrix} m-1 & -1 & \cdots & -1 \\ -1 & m-1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & -1 \\ -1 & \cdots & -1 & m-1 \end{pmatrix}_{m \times m} = (m+4)L(K_m).$$

Therefore, the critical group of $K_m \times C_2$ is $\mathbb{Z}_{m+4} \oplus (\mathbb{Z}_{m(m+4)})^{m-2}$.

(III) If $m = 1$, then $K_m \times C_n$ is just the cycle C_n . So from [14], its critical group is \mathbb{Z}_n . In fact, when $m = 1$, it is easy to see the matrix A in (3.10) has the following property:

$$A = E = \begin{pmatrix} -n & n \\ -n & n \end{pmatrix} \sim (0) \oplus n.$$

The known result is obtained immediately.

(IV) If $m = 2$, the graph $K_2 \times C_n$ is just the Cayley graph \mathcal{D}_n of dihedral group. The result of this case was obtained in [8]. In the following we will try to get the result again.

From (3.10) and (3.13), we know

$$A = \begin{pmatrix} E & F \\ F & E \end{pmatrix} \sim (0) \oplus B \sim \text{diag}(s_1(B), s_2(B), s_3(B), 0).$$

- If $n = 2s + 1$, then from (2.7) we have

$$\begin{cases} s_1(B) = (n, g_s), \\ s_2(B) = h_s, \\ s_3(B) = \frac{nh_s}{(n, g_s)}. \end{cases}$$

Note that $g_s = \frac{n-h_s}{2}$ and n is odd, then $(n, g_s) = (n, 2g_s) = (n, n - h_s) = (n, h_s)$. Therefore,

$$\begin{cases} s_1(B) = (n, h_s), \\ s_2(B) = h_s, \\ s_3(B) = \frac{nh_s}{(n, h_s)}. \end{cases}$$

- If $n = 2s$, then from (2.8) we have

$$\begin{cases} s_1(B) = (u_s, 2\tau_s), \\ s_2(B) = \frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)}, \\ s_3(B) = \frac{6nu_s}{(n, u_s - 4\tau_s)}. \end{cases}$$

Note that $s - u_s$ is even (2.2), and $2^{t+1}|u_n$ if $2^t|n$ (Corollary 3.4 [8]). Hence

$$\begin{aligned} s_1(B) &= (u_s, 2\tau_s) = (u_s, s - u_s) = (u_s, s) \\ &= \begin{cases} (u_s, 2s) = (u_s, n), & \text{if } s \text{ is odd,} \\ \frac{(u_s, 2s)}{2} = \frac{(u_s, n)}{2}, & \text{if } s \text{ is even.} \end{cases} \end{aligned}$$

Also we have $3^t|u_n$ if $3^t|n$ (Corollary 3.4 [8]), then

$$(n, u_s - 2(s - u_s)) = (2s, 3u_s) = (n, u_s).$$

So

$$s_2(B) = \frac{u_s(n, u_s - 4\tau_s)}{(u_s, 2\tau_s)} = \frac{u_s(n, u_s)}{(u_s, 2\tau_s)} = \begin{cases} u_s, & \text{if } s \text{ is odd,} \\ 2u_s, & \text{if } s \text{ is even.} \end{cases}$$

$$\text{Then } s_3(B) = \frac{6nu_s}{(n, u_s - 4\tau_s)} = \frac{6nu_s}{(n, u_s)}.$$

It is easy to see the result of case $m = 2$ here is the same to the result obtained in [8].

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